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Existence of multiple solutions for an elliptic system with sign-changing weight functions

Zonghu Xiu [a,b,*,1], Caisheng Chen [a], Jincheng Huang [a]

[a] College of Science, Hohai University, Nanjing 210098, PR China

[b] Science and Information College, Qingdao Agricultural University, Qingdao 266109, PR China

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ABSTRACT

In this paper, we will consider a class of quasilinear elliptic problem of the form

$$\begin{cases} -div\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right)+g_1(x)|u|^{p-2}u \\ \quad =\dfrac{\alpha}{\alpha+\beta}h(x)|u|^{\alpha-2}u|v|^{\beta}+\lambda H_1(x)|u|^{n-2}u, \\ -div\left(|x|^{-ap}|\nabla v|^{p-2}\nabla v\right)+g_2(x)|v|^{p-2}v \\ \quad =\dfrac{\beta}{\alpha+\beta}h(x)|v|^{\beta-2}v|u|^{\alpha}+\mu H_2(x)|v|^{n-2}v, \\ u(x)>0, \quad v(x)>0, \quad x\in\mathbb{R}^N, \end{cases}$$

where $\lambda, \mu > 0$, $1 < p < N$, $1 < n < p < \alpha+\beta < p^* = \frac{Np}{N-pd}$, $0 \le a < \frac{N-p}{p}$, $a \le b < a+1$, $d = a+1-b > 0$, the weight $g_1(x), g_2(x)$ are bounded and nonnegative functions and $h(x), H_1(x), H_2(x)$ are continuous functions which change sign in $\mathbb{R}^N$. We will prove that the problem has at least two positive solutions by using the Nehari manifold and the fibering maps associated with the Euler function for this problem.

1. Introduction

Recently, Miyagaki and Rodrigues [1] have studied the existence of a positive weak solution to the quasilinear elliptic system with weights

$$\begin{cases} -div(|x|^{-ap}|\nabla u|^{p-2}\nabla u)=\lambda|x|^{-(a+1)+c_1}u^{\alpha}v^{\gamma}, & \text{in } \Omega, \\ -div(|x|^{-bq}|\nabla v|^{q-2}\nabla v)=\lambda|x|^{-(b+1)+c_2}u^{\delta}v^{\beta}, & \text{in } \Omega, \\ u=v=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, with $0 \in \Omega$, $1 < p, q < N$, $0 \le a < \frac{N-p}{p}$, $0 \le b < \frac{N-q}{q}$, $0 \le \alpha < p-1$, $0 \le \beta < q-1$, $\delta, \gamma, c_1, c_2 > 0$ and $\theta = (p-1-\alpha)(q-1-\beta)-\gamma\delta > 0$. By the lower and the upper-solution method, they proved that the problem (1.1) possesses a positive weak solution $(u, v) \in W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$ for each $\lambda > 0$. Similar research can be found in [2–6] and the references therein. Up until now, much attention has been paid to the existence of solutions for the problem (1.1) in a bounded domain. But for the problem (1.1) in an unbounded domain

* Corresponding author at: College of Science, Hohai University, Nanjing 210098, PR China.
E-mail address: qingda@163.com (Z. Xiu).

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Ω or $\mathbb{R}^N$, the existence of multiplicity of solutions has been a delicate question. Many authors studied the existence and asymptotic behavior of the solution for the problem (1.1) with $a = b = 0$ and $\Omega = \mathbb{R}^N$; see [7–10] and the references therein. To the best of our knowledge, little seems to be known about the existence of multiple solutions for problem (1.1) with $a, b \neq 0$ and $\Omega = \mathbb{R}^N$.

In this paper, we are interested in the existence and multiplicity results of positive solutions for the quasilinear elliptic system

$$\begin{cases} -div\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + g_1(x)|u|^{p-2}u = \dfrac{\alpha}{\alpha+\beta}h(x)|u|^{\alpha-2}u|v|^{\beta} + \lambda H_1(x)|u|^{n-2}u, \\ -div\left(|x|^{-ap}|\nabla v|^{p-2}\nabla v\right) + g_2(x)|v|^{p-2}v = \dfrac{\beta}{\alpha+\beta}h(x)|v|^{\beta-2}v|u|^{\alpha} + \mu H_2(x)|v|^{n-2}v, \\ u(x) > 0, \qquad v(x) > 0, \quad x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $\lambda, \mu > 0$, $1 < p < N$, $1 < n < p < \alpha + \beta < p^* = \frac{Np}{N-pd}$, $0 \le a < \frac{N-p}{p}$, $a \le b < a+1$, $d = a+1-b > 0$, the weight $g_1(x), g_2(x)$ are bounded and nonnegative functions with $\|g_1\|_\infty, \|g_2\|_\infty > 0$, and $h(x), H_1(x), H_2(x)$ are continuous functions which change sign in $\mathbb{R}^N$. Throughout the paper, we assume that $h(x), H_1(x), H_2(x)$ satisfy the following conditions.

(A_1) $H_i(x)|x|^{bn} \in L^\theta(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\theta = \frac{p^*}{p^*-n}$, $i = 1, 2$;
(A_2) $h(x)|x|^{b(\alpha+\beta)} \in L^\delta(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\delta = \frac{p^*}{p^*-\alpha-\beta}$.

When $a = 0$, $g_1(x) = g_1(x) = 0$, Hsu [11] considered the following elliptic system in a bounded domain $\Omega \subset \mathbb{R}^N$

$$\begin{cases} -div(|\nabla u|^{p-2}\nabla u) = \lambda|u|^{q-2}u + \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, \quad x \in \Omega, \\ -div(|\nabla v|^{p-2}\nabla v) = \mu|v|^{q-2}v + \dfrac{2\beta}{\alpha+\beta}|v|^{\beta-2}v|u|^{\alpha}, \quad x \in \Omega, \\ u = v = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $\lambda, \mu > 0$, $1 < p < q < N$, $\alpha > 1$, $\beta > 1$ satisfy $\alpha + \beta = p^*$. The author proved that the problem (1.3) has at least one positive solution when $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \lambda_1$, and has at least two positive solutions when $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \lambda_2$.

In this paper, motivated by [1,4,8,12,13,11], we are concerned with the multiplicity results of positive weak solutions of the problem (1.2). We will establish the existence and multiplicity results for the problem (1.2) by the Nehari manifold and variation methods. Since $\Omega = \mathbb{R}^N$ is an unbounded domain, the loss of compactness of the Sobolev embedding renders variational technique more delicate.

In fact, in order to preserve this compactness in our problem (1.2), we introduce a weighted Sobolev space and impose some conditions on the weight functions $h(x)$, $H_1(x)$ and $H_2(x)$. The following Caffarelli–Kohn–Nirenberg inequality [3] will be needed. There exists constant $K_{a,b} > 0$ such that

$$\left(\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*}dx\right)^{\frac{1}{p^*}} \le K_{a,b}\left(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx\right)^{\frac{1}{p}}, \tag{1.4}$$

where $-\infty < a < \frac{N-p}{p}$, $a \le b < a+1$, $d = a+1-b$, and $p^* = \frac{pN}{N-pd}$.

Let $L_b^p(\mathbb{R}^N)$ be the completion of the space $C_0^\infty(\mathbb{R}^N)$ endowed with the norm of

$$\|u\|_{L_b^p} = \left(\int_{\mathbb{R}^N} |x|^{-bp}|u|^p dx\right)^{\frac{1}{p}}, \tag{1.5}$$

and $W_a^{1,p}(\mathbb{R}^N)$ be the completion of the space $C_0^\infty(\mathbb{R}^N)$ endowed with the norm of

$$\|u\|_{W_a^{1,p}} = \left(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx\right)^{\frac{1}{p}}. \tag{1.6}$$

We set

$$H_{1\theta} = \left(\int_{\mathbb{R}^N} (|H_1(x)||x|^{bn})^\theta dx\right)^{\frac{1}{\theta}}, \qquad H_{2\theta} = \left(\int_{\mathbb{R}^N} (|H_2(x) \parallel x|^{bn})^\theta dx\right)^{\frac{1}{\theta}} \tag{1.7}$$

with $\theta = \frac{p*}{p*-n}$.

The natural functional space to study (1.2) is $E = W_a^{1,p}(\mathbb{R}^N) \times W_a^{1,p}(\mathbb{R}^N)$ with respect to the norm

$$\|(u,v)\|_E = \left(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx + g_1(x)|u|^p dx + \int_{\mathbb{R}^N} |x|^{-ap}|\nabla v|^p dx + g_2(x)|v|^p dx\right)^{\frac{1}{p}}. \tag{1.8}$$

Then E is the reflexive Banach space endowed with the norm $\|(u,v)\|_E$.

Definition 1. A pair of functions $(u, v) \in E$ is said to be a weak solution of problem (1.2) if for any $(\phi, \psi) \in E$, there holds

$$\begin{aligned}
&\int_{\mathbb{R}^N} (|x|^{-ap}|\nabla u|^{p-2}\nabla u \nabla \phi + g_1(x)|u|^{p-2}u\phi)dx + \int_{\mathbb{R}^N} (|x|^{-ap}|\nabla v|^{p-2}\nabla v \nabla \psi + g_2(x)|v|^{p-2}v\psi)dx \\
&\quad - \frac{\alpha}{\alpha+\beta}\int_{\mathbb{R}^N} (h(x)|u|^{\alpha-2}u|v|^{\beta} + \lambda H_1(x)|u|^{n-2}u)\phi dx \\
&\quad - \frac{\beta}{\alpha+\beta}\int_{\mathbb{R}^N} (h(x)|v|^{\beta-2}v|u|^{\alpha} + \mu H_2(x)|v|^{n-2}v)\psi dx = 0.
\end{aligned} \tag{1.9}$$

By the assumptions (A_1)–(A_2), all the integrals in (1.9) are well defined and converge.

Our main result is the following.

Theorem 1.1. *Assume that* (A_1) *and* (A_2) *are fulfilled. There exists* $\lambda_0 > 0$ *such that if the parameters* $\lambda, \mu > 0$ *satisfy*

$$0 < \lambda H_{1\theta} + \mu H_{2\theta} < \lambda_0, \tag{1.10}$$

then the problem (1.2) has at least two positive solutions, where $H_{1\theta}, H_{2\theta}$ *are given by (1.7).*

Example. We can choose the following functions which satisfy (A_1)–(A_2) and (1.10) for small $\lambda, \mu > 0$.

$$H_i(x) = \begin{cases} |x|^{k_i} & \text{for } |x| > 1 \; x \in \Omega \\ |x|^{m_i} & \text{for } |x| \le 1 \; x \in \Omega \end{cases}$$

with $k_i < (1-N)/\theta - bn$, $m_i \ge (1-N)/\theta - bn$ $(i = 1, 2)$.

$$h(x) = \begin{cases} |x|^{d_1} & \text{for } |x| > 1 \; x \in \Omega \\ |x|^{d_2} & \text{for } |x| \le 1 \; x \in \Omega \end{cases}$$

with $d_1 < (1-N)/\delta - b(\alpha+\beta)$, $d_2 \ge (1-N)/\delta - b(\alpha+\beta)$.

This paper is organized as follows. In Section 2, we give some properties of the Nehari manifold and set up the variational framework of the problem (1.2). In Section 3, we prove Theorem 1.1. Throughout this paper, we assume (A_1)–(A_2).

2. Preliminaries

It is clear that problem (1.2) has a variational structure. Let $J_{\lambda,\mu}(u, v) : E \to \mathbb{R}^1$ be the corresponding Euler functional of problem (1.2), which is defined by

$$J_{\lambda,\mu}(u, v) = \frac{1}{p}\|(u, v)\|_E^p - \frac{1}{\alpha+\beta}\int_{\mathbb{R}^N} h(x)|u|^{\alpha}|v|^{\beta}dx - \frac{1}{n}G_{\lambda,\mu}(u, v), \tag{2.1}$$

where

$$G_{\lambda,\mu}(u, v) = \int_{\mathbb{R}^N} \lambda H_1(x)|u|^n dx + \int_{\mathbb{R}^N} \mu H_2(x)|v|^n dx. \tag{2.2}$$

Then, we see that the functional $J_{\lambda,\mu} \in C^1(E, \mathbb{R}^1)$ and for $\forall(\phi, \psi) \in E$, there holds

$$\begin{aligned}
\langle J'_{\lambda,\mu}(u, v), (\phi, \psi)\rangle &= \int_{\mathbb{R}^N} (|x|^{-ap}|\nabla u|^{p-2}\nabla u \nabla \phi + g_1|u|^{p-2}u\phi + |x|^{-ap}|\nabla v|^{p-2}\nabla v \nabla \psi + g_2|v|^{p-2}v\psi)dx \\
&\quad - \frac{\alpha}{\alpha+\beta}\int_{\mathbb{R}^N} (h(x)|u|^{\alpha-2}u|v|^{\beta} + \lambda H_1(x)|u|^{n-2}u)\phi dx \\
&\quad - \frac{\beta}{\alpha+\beta}\int_{\mathbb{R}^N} (h(x)|v|^{\beta-2}v|u|^{\alpha} + \mu H_2(x)|v|^{n-2}v)\psi dx.
\end{aligned} \tag{2.3}$$

In particular, it follows from (2.3) that

$$\langle J'_{\lambda,\mu}(u, v), (u, v)\rangle = \|(u, v)\|_E^p - \int_{\mathbb{R}^N} h(x)|u|^{\alpha}|v|^{\beta}dx - G_{\lambda,\mu}(u, v). \tag{2.4}$$

It is well known that the weak solution of problem (1.2) is the critical point of the Euler functional $J_{\lambda,\mu}(u, v)$. Thus, to prove the existence of weak solutions for problem (1.2), it is sufficient to show that $J_{\lambda,\mu}(u, v)$ admits a sequence of critical points.

Since $J_{\lambda,\mu}(u, v)$ is not bounded below on E, it is useful to consider the functional $J_{\lambda,\mu}(u, v)$ on the Nehari manifold

$$M_{\lambda,\mu} = \{(u, v) \in E \setminus (0, 0) | \langle J'_{\lambda,\mu}(u, v), (u, v)\rangle = 0\},$$

where $\langle , \rangle$ denotes the usual duality. Then, $(u, v) \in M_{\lambda,\mu}$ if and only if

$$\|(u, v)\|_E^p - \int_{\mathbb{R}^N} h(x)|u|^\alpha |v|^\beta dx - G_{\lambda,\mu}(u, v) = 0. \tag{2.5}$$

Note that $M_{\lambda,\mu}$ contains every nontrivial weak solution of the problem (1.2) [14].

It is easy to see that if $(u, v) \in M_{\lambda,\mu}$, then

$$J_{\lambda,\mu}(u, v) = \left(\frac{1}{p} - \frac{1}{n}\right) \|(u, v)\|_E^p - \left(\frac{1}{\alpha+\beta} - \frac{1}{n}\right) \int_{\mathbb{R}^N} h(x)|u|^\alpha |v|^\beta dx, \tag{2.6}$$

$$= \left(\frac{1}{p} - \frac{1}{\alpha+\beta}\right) \|(u, v)\|_E^p - \left(\frac{1}{n} - \frac{1}{\alpha+\beta}\right) G_{\lambda,\mu}(u, v). \tag{2.7}$$

Furthermore, we define

$$\Phi_{\lambda,\mu}(u, v) = \langle J'_{\lambda,\mu}(u, v), (u, v)\rangle, \quad \forall (u, v) \in E.$$

Then, for any $(u, v) \in M_{\lambda,\mu}$, we have

$$\begin{aligned}\langle \Phi'_{\lambda,\mu}(u, v), (u, v)\rangle &= p\|(u, v)\|_E^p - (\alpha+\beta) \int_{\mathbb{R}^N} h(x)|u|^\alpha |v|^\beta dx - nG_{\lambda,\mu}(u, v) \\ &= (p-n)\|(u, v)\|_E^p + (n-\alpha-\beta) \int_{\mathbb{R}^N} h(x)|u|^\alpha |v|^\beta dx \\ &= (p-\alpha-\beta)\|(u, v)\|_E^p + (\alpha+\beta-n) G_{\lambda,\mu}(u, v). \end{aligned} \tag{2.8}$$

It is natural to split $M_{\lambda,\mu}$ into three parts:

$$\begin{aligned} M^+_{\lambda,\mu} &= \{(u, v) \in M_{\lambda,\mu} | \langle \Phi'_{\lambda,\mu}(u, v), (u, v)\rangle > 0\}, \\ M^-_{\lambda,\mu} &= \{(u, v) \in M_{\lambda,\mu} | \langle \Phi'_{\lambda,\mu}(u, v), (u, v)\rangle < 0\}, \\ M^0_{\lambda,\mu} &= \{(u, v) \in M_{\lambda,\mu} | \langle \Phi'_{\lambda,\mu}(u, v), (u, v)\rangle = 0\}. \end{aligned} \tag{2.9}$$

We now derive some properties of $M^+_{\lambda,\mu}$, $M^-_{\lambda,\mu}$ and $M^0_{\lambda,\mu}$.

Lemma 2.1. *$J_{\lambda,\mu}(u, v)$ is coercive and bounded below on $M_{\lambda,\mu}$.*

Proof. Since $H_i(x)|x|^{bn} \in L^\theta(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)(i = 1, 2)$, we obtain from the Hölder and Caffarelli–Kohn–Nirenberg inequalities that

$$\begin{aligned}\int_{\mathbb{R}^N} \lambda H_1(x)|u|^n dx &\le \lambda \left(\int_{\mathbb{R}^N} (|H_1(x)||x|^{bn})^\theta dx\right)^{\frac{1}{\theta}} \left(\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*} dx\right)^{\frac{n}{p^*}} \\ &\le \lambda K^n_{a,b} H_{1\theta} \left(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx\right)^{\frac{n}{p}} \\ &\le \lambda K^n_{a,b} H_{1\theta} \|(u, v)\|_E^n. \end{aligned}$$

Similarly, we have

$$\int_{\mathbb{R}^N} \mu H_2(x)|v|^n dx \le \mu K^n_{a,b} H_{2\theta} \|(u, v)\|_E^n.$$

Then,

$$G_{\lambda,\mu}(u, v) \le (\lambda H_{1\theta} + \mu H_{2\theta}) K^n_{a,b} \|(u, v)\|_E^n. \tag{2.10}$$

It follows from (2.1), (2.5) and (2.10) that

$$J_{\lambda,\mu}(u, v) \ge \left(\frac{1}{p} - \frac{1}{\alpha+\beta}\right) \|(u, v)\|_E^p - \left(\frac{1}{n} - \frac{1}{\alpha+\beta}\right) (\lambda H_{1\theta} + \mu H_{2\theta}) K^n_{a,b} \|(u, v)\|_E^n. \tag{2.11}$$

Since $p > n$, the inequality (2.11) shows that $J_{\lambda,\mu}(u, v)$ is coercive and bounded below on $M_{\lambda,\mu}$. Thus, the proof is completed. □

Lemma 2.2. *There exists $\lambda_0 > 0$ such that $M^0_{\lambda,\mu} = \emptyset$ for all λ, μ, which satisfy $0 < \lambda H_{1\theta} + \mu H_{2\theta} < \lambda_0$, where $H_{1\theta}$ and $H_{2\theta}$ are given by (1.7).*

Proof. In fact, we let

$$\lambda_0 = \frac{\alpha+\beta-p}{(\alpha+\beta-n)K_{a,b}^n}\left(\frac{p-n}{(\alpha+\beta-n)2h_\delta K_{a,b}^{\alpha+\beta}}\right)^{(p-n)/(\alpha+\beta-p)}, \tag{2.12}$$

where $\delta = \frac{p*}{p*-\alpha-\beta}$ and

$$h_\delta = \left(\int_{\mathbb{R}}^N |(h(x)||x|^{b(\alpha+\beta)})^\delta dx\right)^{\frac{1}{\delta}} < \infty. \tag{2.13}$$

Suppose otherwise; thus there exist λ and μ which satisfy $0 < \lambda H_{1\theta} + \mu H_{2\theta} < \lambda_0$, such that $M_{\lambda,\mu}^0 \neq \emptyset$, that is, there exists $(u, v) \in M_{\lambda,\mu}^0$. Then, it follows from (2.8) that

$$0 = \langle \Phi'_{\lambda,\mu}(u, v), (u, v)\rangle = (p-\alpha-\beta)\|(u, v)\|_E^p + (\alpha+\beta-n)G_{\lambda,\mu}(u, v) \tag{2.14}$$

$$= (p-n)\|(u, v)\|_E^p + (n-\alpha-\beta)\int_{\mathbb{R}^N} h(x)|u|^\alpha|v|^\beta dx. \tag{2.15}$$

Hence, we obtain from (2.10) and (2.14) that

$$\|(u, v)\|_E \leq \frac{\alpha+\beta-n}{\alpha+\beta-p}[(\lambda H_{1\theta} + \mu H_{2\theta})K_{a,b}^n]^{\frac{1}{p-n}}. \tag{2.16}$$

By (A_2) and the Hölder inequality we have

$$\int_{\mathbb{R}^N} h(x)|u|^{\alpha+\beta}dx \leq \left(\int_{\mathbb{R}^N}(|h(x)||x|^{b(\alpha+\beta)})^\delta dx\right)^{\frac{1}{\delta}}\left(\int_{\mathbb{R}^N}|x|^{-bp^*}|u|^{p^*}dx\right)^{\frac{\alpha+\beta}{p^*}}$$
$$\leq h_\delta K_{a,b}^{\alpha+\beta}\|(u, v)\|_E^{\alpha+\beta}, \tag{2.17}$$

Similarly, we have

$$\int_{\mathbb{R}^N} h(x)|v|^{\alpha+\beta}dx \leq h_\delta K_{a,b}^{\alpha+\beta}\|(u, v)\|_E^{\alpha+\beta}. \tag{2.18}$$

Hence, it follows from (2.17) and (2.18) that

$$\int_{\mathbb{R}^N} h(x)|u|^\alpha|v|^\beta dx \leq 2h_\delta K_{a,b}^{\alpha+\beta}\|(u, v)\|_E^{\alpha+\beta}, \tag{2.19}$$

where $\delta = \frac{p^*}{p^*-\alpha-\beta}$. Therefore, from (2.8) and (2.19) we have

$$\|(u, v)\|_E \geq \left[\frac{p-n}{(\alpha+\beta-n)2h_\delta K_{a,b}^{\alpha+\beta}}\right]^{1/(\alpha+\beta-p)}. \tag{2.20}$$

Then, we obtain from (2.16) and (2.20) that $\lambda H_{1\theta} + \mu H_{2\theta} \geq \lambda_0$, which is a contradiction. □

As the argument in the proof Theorem 1.3 in [15], we have the following lemma.

Lemma 2.3. *Suppose that (u_0, v_0) is a local minimizer for $J_{\lambda,\mu}(u, v)$ on $M_{\lambda,\mu}$, and $(u_0, v_0) \notin M_{\lambda,\mu}^0$. Then (u_0, v_0) is a critical point of $J_{\lambda,\mu}(u, v)$.*

Proof. Let

$$F(u, v) = \|(u, v)\|_E^p - \int_{\mathbb{R}^N} h(x)|u|^\alpha|v|^\beta dx - G_{\lambda,\mu}(u, v).$$

We consider the optimization problem

$$\min_{(u,v)\in M_{\lambda,\mu}} J_{\lambda,\mu}(u, v), \quad \text{subject to } F(u, v) = 0.$$

By the theory of Lagrange multiplier principle, there exists $\eta \in \mathbb{R}^1$ such that

$$J'_{\lambda,\mu}(u_0, v_0) = \eta F'(u_0, v_0).$$

Since $(u_0, v_0) \in M_{\lambda,\mu}$,

$$\langle J'_{\lambda,\mu}(u_0, v_0), (u_0, v_0)\rangle = 0.$$

However, $(u_0, v_0) \notin M^0_{\lambda,\mu}$, thus,

$$\langle F'(u_0, v_0), (u_0, v_0)\rangle = \langle \Phi'_{\lambda,\mu}(u_0, v_0), (u_0, v_0)\rangle \neq 0.$$

Then, $\eta = 0$, and $J'_{\lambda,\mu}(u_0, v_0) = 0$. The proof is completed. □

By Lemma 2.2, we write $M_\lambda = M^+_\lambda \bigcup M^-_\lambda$ for $\lambda \in (0, \lambda_0)$ and define

$$\delta^+_{\lambda,\mu} = \inf_{(u,v)\in M^+_{\lambda,\mu}} J_{\lambda,\mu}(u, v), \qquad \delta^-_{\lambda,\mu} = \inf_{(u,v)\in M^-_{\lambda,\mu}} J_{\lambda,\mu}(u, v). \tag{2.21}$$

Lemma 2.4. *If λ and μ satisfy $0 < \lambda H_{1\theta} + \mu H_{2\theta} < \frac{n\lambda_0}{p}$. Then,*

(i) $\delta^+_{\lambda,\mu} < 0$;
(ii) $\exists k_0 > 0$ *such that* $\delta^-_{\lambda,\mu} > k_0$.

Proof. (i) Let $(u, v) \in M^+_{\lambda,\mu}$. It follows from (2.8) and (2.9) that

$$\int_{\mathbb{R}^N} h(x)|u|^\alpha |v|^\beta dx \le \frac{p-n}{\alpha+\beta-n}\|(u, v)\|^p_E. \tag{2.22}$$

Then, by (2.6) and (2.22) we have that

$$\begin{aligned} J_{\lambda,\mu}(u, v) &\le \left(\frac{1}{p} - \frac{1}{n}\right)\|(u, v)\|^p_E + \left(\frac{1}{n} - \frac{1}{\alpha+\beta}\right)\frac{p-n}{\alpha+\beta-n}\|(u, v)\|^p_E \\ &\le \frac{(p-n)(p-\alpha-\beta)}{np(\alpha+\beta)}\|(u, v)\|^p_E < 0. \end{aligned}$$

Furthermore,

$$\delta^+_{\lambda,\mu} = \inf_{(u,v)\in M^+_{\lambda,\mu}} J_{\lambda,\mu}(u, v) < 0.$$

(ii) Let $(u, v) \in M^-_{\lambda,\mu}$. From (2.7) and (2.10) we have

$$\begin{aligned} J_{\lambda,\mu}(u, v) &= \left(\frac{1}{p} - \frac{1}{\alpha+\beta}\right)\|(u, v)\|^p_E - \left(\frac{1}{n} - \frac{1}{\alpha+\beta}\right)G_{\lambda,\mu}(u, v) \\ &\ge \left(\frac{1}{p} - \frac{1}{\alpha+\beta}\right)\|(u, v)\|^p_E - \left(\frac{1}{n} - \frac{1}{\alpha+\beta}\right)(\lambda H_{1\theta} + \mu H_{2\theta})K^n_{a,b}\|(u, v)\|^n_E \\ &= \|(u, v)\|^n_E\left[\left(\frac{1}{p} - \frac{1}{\alpha+\beta}\right)\|(u, v)\|^{p-n}_E - \left(\frac{\alpha+\beta-n}{n(\alpha+\beta)}\right)(\lambda H_{1\theta} + \mu H_{2\theta})K^n_{a,b}\right]. \end{aligned} \tag{2.23}$$

Thus, it follows from (2.20) and (2.23) that

$$\begin{aligned} J_{\lambda,\mu}(u, v) \ge &\left[\frac{p-n}{(\alpha+\beta-n)2h_\delta K^n_{a,b}}\right]^{\frac{n}{\alpha+\beta-n}}\left[\frac{\alpha+\beta-n}{p(\alpha+\beta)}\left(\frac{p-n}{(\alpha+\beta-n)2h_\delta K^{\alpha+\beta}_{a,b}}\right)^{\frac{p-n}{\alpha+\beta-p}}\right. \\ &\left. - \frac{\alpha+\beta-n}{n(\alpha+\beta)}(\lambda H_{1\theta} + \mu H_{2\theta})K^n_{a,b}\right]. \end{aligned}$$

If

$$0 < \lambda H_{1\theta} + \mu H_{2\theta} < \frac{n(\alpha+\beta-p)}{p(\alpha+\beta-n)K^n_{a,b}}\left[\frac{p-n}{(\alpha+\beta-n)2h_\delta K^{\alpha+\beta}_{a,b}}\right]^{\frac{p-n}{\alpha+\beta-p}} = \frac{n\lambda_0}{p},$$

then, there exists $k_0(\alpha, \beta, p, n, h_\delta, K_{a,b}) > 0$ such that $\delta^-_{\lambda,\mu} > k_0$. □

For each $(u, v) \in E$ with $\int_{\mathbb{R}^N} h(x)|u|^\alpha |v|^\beta dx > 0$ we set

$$T_0 = \left[\frac{(p-n)\|(u, v)\|^p_E}{(\alpha+\beta-n)\int_{\mathbb{R}^N} h(x)|u|^\alpha |v|^\beta dx}\right]^{1/(\alpha+\beta-p)},$$

then, we have the following results.

Lemma 2.5. *Assume* $\int_{\mathbb{R}^N} h(x)|u|^\alpha|v|^\beta dx > 0$ *and* $0 < \lambda H_{1\theta} + \mu H_{2\theta} < \lambda_0$. *Then,*

(i) *if* $G_{\lambda,\mu}(u, v) \le 0$, *there exists unique* $t^- > T_0$ *such that* $(t^-u, t^-v) \in M^-_{\lambda,\mu}$ *and*

$$J_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\ge 0} J_{\lambda,\mu}(tu, tv);$$

(ii) *if* $G_{\lambda,\mu}(u, v) > 0$, *there exist* $0 < t^+ < T_0 < t^-$ *such that* $(t^+u, t^+v) \in M^+_{\lambda,\mu}$, $(t^-u, t^-v) \in M^-_{\lambda,\mu}$ *and*

$$J_{\lambda,\mu}(t^+u, t^+v) = \inf_{0\le t\le T_0} J_{\lambda,\mu}(tu, tv), J_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\ge 0} J_{\lambda,\mu}(tu, tv).$$

Proof. Set

$$\begin{aligned}\varphi_0(t) &= \langle J'_{\lambda,\mu}(tu, tv), (tu, tv)\rangle \\ &= t^p\|(u, v)\|_E^p - t^{\alpha+\beta}\int_{\mathbb{R}^N} h(x)|u|^\alpha|v|^\beta dx - t^n G_{\lambda,\mu}(u, v), \end{aligned} \tag{2.24}$$

$$\begin{aligned}\varphi_1(t) &= \langle \Phi'_{\lambda,\mu}(tu, tv), (tu, tv)\rangle \\ &= pt^p\|(u, v)\|_E^p - (\alpha+\beta)t^{\alpha+\beta}\int_{\mathbb{R}^N} h(x)|u|^\alpha|v|^\beta dx - nt^n G_{\lambda,\mu}(u, v), \end{aligned} \tag{2.25}$$

$$\begin{aligned}\varphi_2(t) &= J_{\lambda,\mu}(tu, tv) \\ &= \frac{t^p}{p}\|(u, v)\|_E^p - \frac{t^{\alpha+\beta}}{\alpha+\beta}\int_{\mathbb{R}^N} h(x)|u|^\alpha|v|^\beta dx - \frac{t^n}{n} G_{\lambda,\mu}(u, v), \end{aligned} \tag{2.26}$$

and

$$m(t) = t^{p-n}\|(u, v)\|_E^p - t^{\alpha+\beta-n}\int_{\mathbb{R}^N} h(x)|u|^\alpha|v|^\beta dx.$$

Then,

$$\varphi_0(t) = t^n[m(t) - G_{\lambda,\mu}(u, v)].$$

Since $m(t) \to 0$ as $t \to 0^+$ and $m(t) \to -\infty$ as $t \to \infty$, $m(t)$ has unique critical point at T_0.

It is easy to check that $m(t)$ increases in $[0, T_0]$, and decreases in $[T_0, +\infty)$.

(i) $G_{\lambda,\mu}(u, v) < 0$.

There exists unique $t^- > T_0$ such that $m(t^-) = G_{\lambda,\mu}(u, v)$. It follows from $\varphi_0(t^-) = 0$ that $(t^-u, t^-v) \in M_{\lambda,\mu}$. Then, from (2.25) we get that $\varphi_1(t^-) = (t^-)^{n+1}m'(t^-) < 0$, which implies that $(t^-u, t^-v) \in M^-_{\lambda,\mu}$. By simple calculation, we obtain that $\varphi_2'(t) = t^{n-1}[m(t) - G_{\lambda,\mu}(u, v)]$. Furthermore, $\varphi_2'(t) > 0$ for $t \in [0, t^-]$ and $\varphi_2'(t) < 0$ for $t \in [t^-, +\infty)$. Then, $\varphi_2(t)$ gets its maximum at t^-, that is

$$J_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\ge 0} J_{\lambda,\mu}(tu, tv).$$

(ii) $G_{\lambda,\mu}(u, v) > 0$.

Since $0 < \lambda H_{1\theta} + \mu H_{2\theta} < \lambda_0$, by (2.10) we get that

$$\begin{aligned} m(0) &= 0 \\ &< G_{\lambda,\mu}(u, v) \le (\lambda H_{1\theta} + \mu H_{2\theta})K^n_{a,b}\|(u, v)\|_E^n \\ &< \left(\frac{\alpha+\beta-p}{\alpha+\beta-n}\right)\left[\frac{p-n}{(\alpha+\beta-n)2h_\delta K^{\alpha+\beta}_{a,b}}\right]^{\frac{p-n}{\alpha+\beta-p}}\|(u, v)\|_E^n \\ &\le m(T_0). \end{aligned}$$

Then, there exist t^+ and t^- such that $0 < t^+ < T_0 < t^-$ and $m(t^+) = m(t^-) = G_{\lambda,\mu}(u, v)$. Similar to the argument in (i), we have $(t^+u, t^+v) \in M^+_{\lambda,\mu}$ and $(t^-u, t^-v) \in M^-_{\lambda,\mu}$. Since $\varphi_2'(t) < 0$ for $t \in [0, t^+]$ and $\varphi_2'(t) > 0$ for $t \in [t^+, T_0]$, $J_{\lambda,\mu}(t^+u, t^+v) = \inf_{0\le t\le T_0} J_{\lambda,\mu}(tu, tv)$. Furthermore, it is easy to find that $\varphi_2'(t) > 0$ for $t \in [t^+, t^-]$, $\varphi_2'(t) < 0$ for $t \in [t^-, +\infty]$ and $\varphi_2(t) \le 0$ for $t \in [0, t^+]$. Furthermore, since $(t^-u, t^-v) \in M^-_{\lambda,\mu}$, by Lemma 2.4 we have $\varphi_2(t^-) > 0$. Then, $J_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\ge 0} J_{\lambda,\mu}(tu, tv)$. □

Similar to Lemma 2.5, for each $(u, v) \in E$ with $G_{\lambda,\mu}(u, v) > 0$, we set

$$T_1(u, v) = \left[\frac{(\alpha+\beta-n)G_{\lambda,\mu}(u, v)}{(\alpha+\beta-p)\|(u, v)\|_E^p}\right]^{1/(p-n)}.$$

Then, we have the following lemma.

Lemma 2.6. *For each $(u, v) \in E$ with $G_{\lambda,\mu}(u, v) > 0$ and $0 < \lambda H_{1\theta} + \mu H_{2\theta} < \lambda_0$,*

(i) *if $\int_{\mathbb{R}^N} h(x)|u|^\alpha |v|^\beta dx \le 0$, then, there exists unique $t^+ < T_1$ such that $(t^+u, t^+v) \in M^+_{\lambda,\mu}$ and $J_{\lambda,\mu}(t^+u, t^+v) = \inf_{0\le t\le T_1} J_{\lambda,\mu}(tu, tv)$;*

(ii) *if $\int_{\mathbb{R}^N} h(x)|u|^\alpha |v|^\beta dx > 0$, there exist $0 < t^+ < T_1 < t^-$ such that $(t^+u, t^+v) \in M^+_{\lambda,\mu}$, $(t^-u, t^-v) \in M^-_{\lambda,\mu}$ and*

$$J_{\lambda,\mu}(t^+u, t^+v) = \inf_{0\le t\le T_1} J_{\lambda,\mu}(tu, tv), \qquad J_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\ge 0} J_{\lambda,\mu}(tu, tv).$$

Proof. Let

$$m_1(t) = t^{p-\alpha-\beta}\|(u, v)\|_E^p - t^{n-\alpha-\beta} G_{\lambda,\mu}(u, v), \quad t > 0.$$

Clearly, $m_1(t) \to -\infty$ as $t \to 0^+$; $m_1(t) \to 0$ as $t \to +\infty$ and $m_1'(t) > 0$ for $t \in (0, T_1]$, $m_1'(t) < 0$ for $t \in [T_1, +\infty)$. Then, $m_1(t)$ obtain its maximum at $t = T_1$. When $\int_{\mathbb{R}^N} h(x)|u|^\alpha |v|^\beta dx > 0$ we can also prove that $0 < \int_{\mathbb{R}^N} h(x)|u|^\alpha |v|^\beta dx < m_1(T_1)$. Similar to the proof of Lemma 2.5, Lemma 2.6 can be proved. □

Lemma 2.7. *Assume (A_1)–(A_2). If $u_k \rightharpoonup u_0$, $v_k \rightharpoonup v_0$ weakly in E, then there exists a subsequence of $\{(u_k, v_k)\}$, still denoted by $\{(u_k, v_k)\}$, such that*

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} h(x)|u_k|^\alpha |v_k|^\beta dx = \int_{\mathbb{R}^N} h(x)|u_0|^\alpha |v_0|^\beta dx, \tag{2.27}$$

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} H_1(x)|u_k|^n dx = \int_{\mathbb{R}^N} H_1(x)|u_0|^n dx, \tag{2.28}$$

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} H_2(x)|v_k|^n dx = \int_{\mathbb{R}^N} H_2(x)|v_0|^n dx. \tag{2.29}$$

Proof. We only prove (2.27), and (2.28)–(2.29) can be similarly obtain as the proof of (2.27). From (A_2) we have that for any $\varepsilon > 0$ there exists $R_0 > 0$ such that

$$\int_{B^c_{R_0}} (|h(x)||x|^{b(\alpha+\beta)})^\delta dx < \varepsilon^\delta$$

where $B_r = \{x \in \mathbb{R}^N \parallel x| \le r\}$ and $B^c_r = \{x \in \mathbb{R}^N \parallel x| > r\}$ for any $r > 0$. Since $u_k \rightharpoonup u_0$, $v_k \rightharpoonup v_0$ weakly in E, $\{(u_k, v_k)\}$ is bounded in E and $u_k \rightharpoonup u_0$, $v_k \rightharpoonup v_0$ weakly in $W^{1,p}_a(\mathbb{R}^N)$. Furthermore, (1.4) means that $\{(u_k, v_k)\}$ is bounded in $L^{p^*}_b(\mathbb{R}^N)$. Then, we have that

$$u_k \rightharpoonup u_0, \qquad v_k \rightharpoonup v_0 \quad \text{in } L^{p^*}_{b,loc}(\mathbb{R}^N \setminus \{0\}), \tag{2.30}$$

and

$$u_k \to u_0, \qquad v_k \to v_0 \quad \text{a.e. in } \mathbb{R}^N. \tag{2.31}$$

It follows from (2.30) and (2.31) that for large k

$$\int_{B_{R_0}} |x|^{-bp^*}|u_k - u_0|^{p^*} < \varepsilon^{p^*}, \tag{2.32}$$

and for $k \ge 1$ there exists $M > 0$, which is independent of k such that

$$\int_{\mathbb{R}^N} |x|^{-bp^*}|u_k|^{p^*} dx \le M^{p^*}, \qquad \int_{\mathbb{R}^N} |x|^{-bp^*}|u_0|^{p^*} dx \le M^{p^*}. \tag{2.33}$$

Similarly, from (2.30) and (2.31) we have that for large k

$$\int_{B_{R_0}} |x|^{-bp^*}|v_k - v_0| < \varepsilon^{p^*}.$$

Furthermore, we get from the Hölder inequality that

$$\begin{aligned} \int_{B^c_{R_0}} h(x)|u_k - u_0|^{\alpha+\beta} dx &\le \left(\int_{B^c_{R_0}} (|h(x)||x|^{b(\alpha+\beta)})^\delta dx\right)^{\frac{1}{\delta}} \left(\int_{B^c_{R_0}} |x|^{-bp^*}|u_k - u_0|^{p^*} dx\right)^{\frac{\alpha+\beta}{p^*}} \\ &\le 2^{\alpha+\beta} M^{\alpha+\beta}\varepsilon. \end{aligned} \tag{2.34}$$

By (A_2) and for large k we have

$$\int_{B_{R_0}} h(x)|u_k-u_0|^{\alpha+\beta}dx \le \left(\int_{B_{R_0}} (|h(x)||x|^{b(\alpha+\beta)})^{\delta}dx\right)^{\frac{1}{\delta}} \left(\int_{B_{R_0}} |x|^{-bp^*}|u_k-u_0|^{p^*}dx\right)^{\frac{\alpha+\beta}{p^*}}$$
$$\le M_0\varepsilon^{\alpha+\beta} \tag{2.35}$$

for some $M_0>0$. Thus, we obtain from (2.34) and (2.35) that

$$\lim_{k\to\infty}\int_{\mathbb{R}^N} h(x)|u_k-u_0|^{\alpha+\beta}dx=0.$$

Similarly, we obtain

$$\lim_{k\to\infty}\int_{\mathbb{R}^N} h(x)|v_k-v_0|^{\alpha+\beta}dx=0.$$

Note that

$$\int_{\mathbb{R}^N}|h(x)\|u_k-u_0|^{\alpha}|v_k-v_0|^{\beta}dx \le \int_{\mathbb{R}^N}|h(x)\|u_k-u_0|^{\alpha+\beta}dx+\int_{\mathbb{R}^N}|h(x)||v_k-v_0|^{\alpha+\beta}dx,$$

then we get (2.27). This gives the proof. □

3. Existence of positive solutions

Lemma 3.1. *If* $0<\lambda H_{1\theta}+\mu H_{2\theta}<\frac{n\lambda_0}{p}$, *then, there is a minimizer* $(u_0^+,v_0^+)\in M_{\lambda,\mu}^+$ *such that*

(i) $J_{\lambda,\mu}(u_0^+,v_0^+)=\delta_{\lambda,\mu}^+$;
(ii) (u_0^+,v_0^+) *is a positive solution of problem* (1.2).

Proof. By Lemma 2.1, note that $J_{\lambda,\mu}(u,v)$ is bounded on $M_{\lambda,\mu}^+$; then there exists a minimizing sequence $\{(u_k,v_k)\}\subseteq M_{\lambda,\mu}^+$ such that

$$\lim_{k\to\infty}J_{\lambda,\mu}(u_k,v_k)=\inf_{(u,v)\in M_{\lambda,\mu}^+}J_{\lambda,\mu}(u,v).$$

Since $J_{\lambda,\mu}(u,v)$ is coercive, $\{(u_k,v_k)\}$ is bounded in E. Thus, we may assume that

$$u_k\rightharpoonup u_0^+,\ u_k\rightharpoonup v_0^+,$$

in E. By Lemma 2.4, we have

$$J_{\lambda,\mu}(u_k,v_k)\to\delta_{\lambda,\mu}^+<0.$$

Let $k\to\infty$, it follows from Lemma 2.4, (2.7), (2.28) and (2.29) that $G_{\lambda,\mu}(u_0^+,v_0^+)>0$. In the following, we prove that

$$u_k\to u_0^+,\qquad v_k\to v_0^+$$

in E. Suppose otherwise; then

$$\|u_0^+\|_{W_a^{1,p}}<\lim_{k\to\infty}\inf\|u_k\|_{W_a^{1,p}}\quad\text{or}\quad\|v_0^+\|_{W_a^{1,p}}<\lim_{k\to\infty}\inf\|v_k\|_{W_a^{1,p}}. \tag{3.1}$$

Let

$$m_2(t)=t^{p-\alpha-\beta}\|(u_0^+,v_0^+)\|_E^p-t^{n-\alpha-\beta}G_{\lambda,\mu}(u_0^+,v_0^+),$$

then, $m_2(t)$ has a unique critical point at

$$T_2\triangleq T_1(u_0^+,v_0^+)=\left[\frac{(\alpha+\beta-n)G_{\lambda,\mu}(u_0^+,v_0^+)}{(\alpha+\beta-p)\|(u_0^+,v_0^+)\|_E^p}\right]^{\frac{1}{p-n}},$$

and $m_2'(t)>0$ for $t\in(0,T_2]$, $m_2'(t)<0$ for $t\in[T_2,+\infty)$. This implies that $m_2(t)$ has its maximum at T_2. Let

$$\phi_0(t)=m_2(t)-\int_{\mathbb{R}^N}h(x)|u_0^+|^{\alpha}|v_0^+|^{\beta}dx.$$

Then, $\phi_0(t)\to-\infty$ as $t\to0$ and $\phi_0(t)\to-\int_{\mathbb{R}^N}h(x)|u_0^+|^{\alpha}|v_0^+|^{\beta}dx$ as $t\to+\infty$.

Set

$$\psi_0(t) = J_{\lambda,\mu}(tu_0^+, tv_0^+) = \frac{t^p}{p}\|(u_0^+, v_0^+)\|_E^p - \frac{t^{\alpha+\beta}}{\alpha+\beta}\int_{\mathbb{R}^N} h(x)|u_0^+|^\alpha|v_0^+|^\beta dx - \frac{t^n}{n}G_{\lambda,\mu}(u_0^+, v_0^+).$$

Thus,

$$\psi_0'(t) = t^{\alpha+\beta-1}\left[m_2(t) - \int_{\mathbb{R}^N} h(x)|u_0^+|^\alpha|v_0^+|^\beta dx\right].$$

By Lemma 2.6, there exists unique $t_0^+ < T_2$ such that $m_2'(t_0^+) > 0$, $m_2(t_0^+) = \int_{\mathbb{R}^N} h(x)|u_0^+|^\alpha|v_0^+|^\beta dx$, $(t_0^+u_0^+, t_0^+v_0^+) \in M_{\lambda,\mu}^+$ and $J_{\lambda,\mu}(t_0^+u_0^+, t_0^+v_0^+) = \inf_{0\le t\le T_2} J_{\lambda,\mu}(tu_0^+, tv_0^+)$. Furthermore,

$$\begin{aligned}\phi_0(t_0^+) &= (t_0^+)^{p-\alpha-\beta}\|(u_0^+, v_0^+)\|_E^p - (t_0^+)^{n-\alpha-\beta}G_{\lambda,\mu}(u_0^+, v_0^+) - \int_{\mathbb{R}^N} h(x)|u_0^+|^\alpha|v_0^+|^\beta dx,\\ &= (t_0^+)^{-\alpha-\beta}\langle J_{\lambda,\mu}'(t_0^+u_0^+, t_0^+v_0^+), (t_0^+u_0^+, t_0^+v_0^+)\rangle = 0.\end{aligned}$$

For the further proof, we set

$$\phi_k(t) = t^{p-\alpha-\beta}\|(u_k, v_k)\|_E^p - t^{n-\alpha-\beta}G_{\lambda,\mu}(u_k, v_k) - \int_{\mathbb{R}^N} h(x)|u_k|^\alpha|v_k|^\beta dx.$$

By (3.1), we have $\phi_k(t_0^+) > \phi_0(t_0^+) = 0$ for large k. Since $\langle \Phi_{\lambda,\mu}'(u_k, v_k), (u_k, v_k)\rangle > 0$,

$$(p-\alpha-\beta)\|(u_k, v_k)\|_E^p + (\alpha+\beta-n)G_{\lambda,\mu}(u_k, v_k) > 0,$$

which implies that $T_1(u_k, v_k) > 1$; furthermore, $T_2 = T_1(u_0^+, v_0^+) > 1$. Since $\phi_k(1) = 0$ and $\phi_k'(t) > 0$ for $t \in [0, T_2]$, $\phi_k(t) \le 0$ for all $t \in [0, 1]$. From $\phi_k(t_0^+) > 0$, we have $t_0^+ > 1$. Thus, $1 < t_0^+ < T_2$ and

$$J_{\lambda,\mu}(t_0^+u_0^+, t_0^+v_0^+) \le J_{\lambda,\mu}(u_0^+, v_0^+) < \lim_{k\to\infty} J_{\lambda,\mu}(u_k, v_k) = \delta_{\lambda,\mu}^+,$$

which is a contradiction. Hence, $u_k \to u_0^+$, $v_k \to v_0^+$ strongly in $W_a^{1,p}(\mathbb{R}^N)$. This implies that

$$J_{\lambda,\mu}(u_k, v_k) \to J_{\lambda,\mu}(u_0^+, v_0^+) = \delta_{\lambda,\mu}^+ \quad \text{as } t \to \infty.$$

Thus, (u_0^+, v_0^+) is a minimizer of $J_{\lambda,\mu}(u, v)$ on $M_{\lambda,\mu}^+$. Since $J_{\lambda,\mu}(u_0^+, v_0^+) = J_{\lambda,\mu}(|u_0^+|, |v_0^+|)$ and $(|u_0^+|, |v_0^+|) \in M_{\lambda,\mu}^+$, by Lemma 2.3, we may assume that (u_0^+, v_0^+) is a nonnegative solution of problem (1.2). Furthermore, we obtain that $u_0^+ > 0$, $v_0^+ > 0$ by the maximum principle; see [2,16]. This concludes the proof. □

Lemma 3.2. *Assume* $0 < \lambda H_{1\theta} + \mu H_{2\theta} < \frac{n\lambda_0}{p}$. *Then,* $J_{\lambda,\mu}(u, v)$ *has minimizer* $(u_0^-, v_0^-) \in M_{\lambda,\mu}^-$ *and*

(i) $J_{\lambda,\mu}(u_0^-, v_0^-) = \delta_{\lambda,\mu}^-$;
(ii) (u_0^-, v_0^-) *is a positive solution of problem* (1.2).

Proof. It follows from Lemma 2.1 that $J_{\lambda,\mu}(u, v)$ is coercive and bounded below on $M_{\lambda,\mu}^-$. Then, there exists a minimizing sequence $\{(u_k, v_k)\} \subseteq M_{\lambda,\mu}^-$ such that

$$\lim_{k\to\infty} J_{\lambda,\mu}(u_k, v_k) = \inf_{(u,v)\in M_{\lambda,\mu}^-} J_{\lambda,\mu}(u, v).$$

Since $J_{\lambda,\mu}(u, v)$ is coercive, $\{(u_k, v_k)\}$ is bounded in E. Therefore, there exists a subsequence of $\{(u_k, v_k)\}$, still denoted by $\{(u_k, v_k)\}$, and $(u_0^-, v_0^-) \in E$ such that $\{(u_k, v_k)\} \rightharpoonup (u_0^-, v_0^-)$. By Lemma 2.4, we have that $J_{\lambda,\mu}(u, v) > 0$ for all $(u, v) \in M_{\lambda,\mu}^-$. Thus

$$\inf_{(u,v)\in M_{\lambda,\mu}^-} J_{\lambda,\mu}(u, v) > 0.$$

Furthermore, by (2.6) we obtain that $\int_{\mathbb{R}^N} h(x)|u_k|^\alpha|v_k|^\beta dx > 0$. Therefore, it follows from Lemma 2.7 that

$$\int_{\mathbb{R}^N} h(x)|u_0^-|^\alpha|v_0^-|^\beta dx > 0. \tag{3.2}$$

Now, we prove that $u_k \to u_0^-$, $v_k \to v_0^-$ strongly in $W_a^{1,p}(\mathbb{R}^N)$. Suppose otherwise; then

$$\|u_0^-\|_{W_a^{1,p}} < \lim_{k\to\infty}\inf\|u_k\|_{W_a^{1,p}} \quad \text{or} \quad \|v_0^-\|_{W_a^{1,p}} < \lim_{k\to\infty}\inf\|v_k\|_{W_a^{1,p}}.$$

Since $\int_{\mathbb{R}^N} h(x)|u_0^-|^\alpha |v_0^-|^\beta dx > 0$, it follows from Lemma 2.5 that there exists unique t_0^- such that $(t_0^- u_0^-, t_0^- v_0^-) \in M_{\lambda,\mu}^-$. For any $(u_k, v_k) \in M_{\lambda,\mu}^-$, Lemma 2.5 and a simple transformation imply that $J_{\lambda,\mu}(u_k, v_k) > J_{\lambda,\mu}(tu_k, tv_k)$ for all $t \geq 0$. Then

$$J_{\lambda,\mu}(t_0^- u_0^-, t_0^- v_0^-) < \lim_{k\to\infty} J_{\lambda,\mu}(t_0^- u_k, t_0^- v_k) \leq \lim_{k\to\infty} J_{\lambda,\mu}(u_k, v_k) = \delta_{\lambda,\mu}^-.$$

This is a contradiction; hence, $u_k \to u_0^-$, $v_k \to v_0^-$ strongly in $W_a^{1,p}(\mathbb{R}^N)$, which implies that $J_{\lambda,\mu}(u_k, v_k) \to J_{\lambda,\mu}(u_0^-, v_0^-) = \delta_{\lambda,\mu}^-$ as $k \to \infty$. Since $J_{\lambda,\mu}(u_0^-, v_0^-) = J_{\lambda,\mu}(|u_0^-|, |v_0^-|)$ and $(|u_0^-|, |v_0^-|) \in M_{\lambda,\mu}^-$, similar to the argument in Lemma 3.1, we can also get that (u_0^-, v_0^-) is a positive solution of problem (1.2). □

Proof of Theorem 1.1. By Lemmas 3.1 and 3.2, we obtain that the problem (1.2) has two positive solutions $(u_0^+, v_0^+) \in M_{\lambda,\mu}^+$ and $(u_0^-, v_0^-) \in M_{\lambda,\mu}^-$. Since $M_{\lambda,\mu}^+ \bigcap M_{\lambda,\mu}^- = \emptyset$, the solutions (u_0^+, v_0^+) and (u_0^-, v_0^-) are distinct. Then we complete the proof of Theorem 1.1. □

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